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Synchronisation of Autonomous Agents with Individual Dynamics

How to determine the initial state of the virtual reference system

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Synchronisation of Autonomous Agents with Individual Dynamics

How to determine the initial state of the virtual reference system

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6. Februar 2022

Abstract. The task of synchronising autonomous agents is solved by a networked controller that steers the agents towards a common trajectory. The Internal-Reference Principle says that the agents can be synchronised by an appropriate networked controller only if their dynamics intersect. This intersection acts as the virtual reference system and generates the synchronous trajectory. The report shows two ways to find the initial state of the virtual reference system in dependence upon the initial agent states.

Contents

1 Introduction

A synchronised multi-agent system satisfies the following two requirements:

1. **Synchronous behaviour:** For specific initial states of the agents, all outputs $y_i(t)$ follow a common trajectory $y_{\rm s}(t)$
$$y_1(t) = y_2(t) = ... = y_N(t) = y_{\rm s}(t) \quad t \geq 0, \tag{1}$$
which is called the synchronous trajectory.

2. **Asymptotic synchronisation:** For all other initial states, the networked controller asymptotically synchronises the agents:
$$\lim_{t\to\infty} \|y_i(t) - y_{\rm s}(t)\| = 0, \quad i = 1, 2, ..., N. \tag{2}$$

The synchronous trajectory $y_{\rm s}(t)$ is generated by the virtual reference system introduced in Section 2. The initial state $\boldsymbol{x}_{\rm s0}$ of this system depends upon the initial states $\boldsymbol{x}_{i0}$, $(i = 1, 2, .., N)$ of all agents. This report deals with the problem to determine $\boldsymbol{x}_{\rm s0}$. For comparison, this initial state is obtained for consensus problems of integrator systems as
$$x_{\rm s0} = \sum_{i=1}^{N} \hat{w}_i x_{i0}$$
(eqn. (3.20) on p. 72 in [1]) and for the synchronisation of identical agents as
$$\boldsymbol{x}_{\rm s0} = \sum_{i=1}^{N} \hat{w}_i \boldsymbol{x}_{i0}$$
(eqn. (4.34) on p. 143) with $\hat{w}_i$ denoting the i-th element of the left eigenvector $\boldsymbol{w}^{\rm T}$ of the Laplacian matrix $\boldsymbol{L}$ for the vanishing eigenvalue.

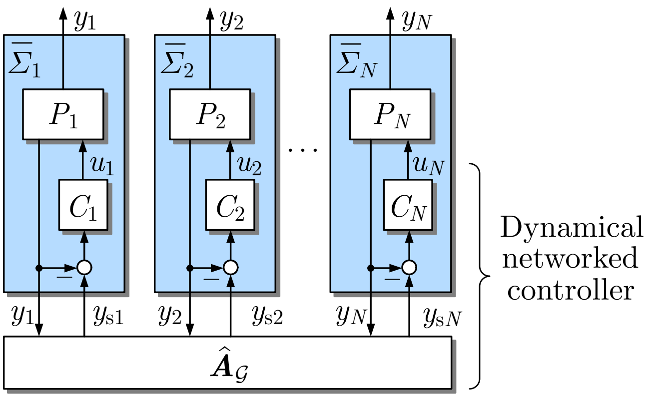

Fig. 1: Multi-agent system with dynamical networked controller

The report refers directly to the results in [1] by using the enumeration of the equations and theorems of the textbook (e. g. Theorem 4.5 or eqn. (4.116)).

2 Models

2.1 Agents with individual dynamics

The agents P_i have linear dynamics with individual parameters

$$P_i: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_i(t) & = & \boldsymbol{A}_i \boldsymbol{x}_i(t) + \boldsymbol{b}_i u_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ y_i(t) & = & \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{x}_i(t) \end{array} \right. \tag{3}$$

$(i = 1, 2, ..., N)$ with

- $u_i(t)$ – scalar input,
- $\boldsymbol{x}_i(t)$ – n_i-dimensional state vector,
- $y_i(t)$ – scalar output.

They are assumed to be completely controllable and completely observable.

The virtual reference system Σ_{s} is an autonomous system

$$\Sigma_{\mathrm{s}}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\mathrm{s}}(t) & = & \boldsymbol{A}_{\mathrm{s}} \boldsymbol{x}_{\mathrm{s}}(t), \quad \boldsymbol{x}_{\mathrm{s}}(0) = \boldsymbol{x}_{\mathrm{s}0} \\ y_{\mathrm{s}}(t) & = & \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \boldsymbol{x}_{\mathrm{s}}(t) \end{array} \right. \tag{4}$$

with the n_{s}-dimensional state $\boldsymbol{x}_{\mathrm{s}}(t)$ and the scalar output $y_{\mathrm{s}}(t)$. It parameterises the set of synchronous trajectories $y_{\mathrm{s}}(t)$ with respect to the initial state $\boldsymbol{x}_{\mathrm{s}0}$:

$$\mathcal{Y}_{\mathrm{s}} = \left\{ \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \mathrm{e}^{\boldsymbol{A}_{\mathrm{s}} t} \boldsymbol{x}_{\mathrm{s}0} \,|\, \boldsymbol{x}_{\mathrm{s}0} \in \mathrm{IR}^{n_{\mathrm{s}}} \right\}.$$

Asymptotic synchronisation (2) means to bring all agents on the same trajectory $y_{\mathrm{s}}(t) \in \mathcal{Y}_{\mathrm{s}}$.

2.2 Networked controller

The generalised synchronisation error of the i-th agent is defined as a linear combination of the output differences

$$e_i(t) = -\sum_{j=1, j\neq i}^{N} l_{ij}(y_i(t) - y_j(t)) = -\sum_{j=1}^{N} l_{ij} y_j(t), \quad i = 1, 2, ..., N \tag{5}$$

with

$$l_{ii} = -\sum_{j=1, j\neq i}^{N} l_{ij}.$$

It can be written in matrix-vector form as

$$\boldsymbol{e}(t) = -\boldsymbol{L}\boldsymbol{y}(t) \tag{6}$$

with the vectors

$$\boldsymbol{e}(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_N(t) \end{pmatrix} \quad \text{and} \quad \boldsymbol{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{pmatrix}$$

and the Laplacian matrix $\boldsymbol{L} = (l_{ij})$.

Each local controller C_i is a dynamical system with an $n_{\mathrm{r}i}$-dimensional internal state $\boldsymbol{x}_{\mathrm{r}i}(t)$:

$$C_i: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{\mathrm{r}i}(t) & = & \boldsymbol{A}_{\mathrm{r}i}\boldsymbol{x}_{\mathrm{r}i}(t) + \boldsymbol{b}_{\mathrm{r}i}e_i(t), \quad \boldsymbol{x}_{\mathrm{r}i}(0) = \boldsymbol{x}_{\mathrm{r}i0} \\ u_i(t) & = & \boldsymbol{k}_{\mathrm{r}i}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{r}i}(t) + k_{\mathrm{e}i}e_i(t). \end{array} \right. \tag{7}$$

It feeds the synchronisation error $e_i(t)$ locally back to the input $u_i(t)$ of the corresponding agent.

In summary, the networked controller is composed of the local controllers C_i of the agents and of the communication network described by the Laplacian matrix $\boldsymbol{L}$ (cf. Fig. 1). Static networked controllers do not include local controllers and the relation

$$C_i: \quad u_i(t) = e_i(t)$$

replaces eqn. (7).

2.3 Extended agents

If the agent model (3) is combined with the local controller (7), the model of the series connection of both components is obtained

$$\begin{aligned} \begin{pmatrix} \dot{\boldsymbol{x}}_i(t) \\ \dot{\boldsymbol{x}}_{\mathrm{r}i}(t) \end{pmatrix} &= \underbrace{\begin{pmatrix} \boldsymbol{A}_i & \boldsymbol{b}_i\boldsymbol{k}_{\mathrm{r}i}^{\mathrm{T}} \\ \boldsymbol{O} & \boldsymbol{A}_{\mathrm{r}i} \end{pmatrix}}_{\boldsymbol{A}_{0i}} \underbrace{\begin{pmatrix} \boldsymbol{x}_i(t) \\ \boldsymbol{x}_{\mathrm{r}i}(t) \end{pmatrix}}_{\boldsymbol{x}_{0i}(t)} + \underbrace{\begin{pmatrix} \boldsymbol{b}_i k_{\mathrm{e}i} \\ \boldsymbol{b}_{\mathrm{r}i} \end{pmatrix}}_{\boldsymbol{b}_{0i}} e_i(t) \\ y_i(t) &= \underbrace{(\boldsymbol{c}_i^{\mathrm{T}} \;\; \boldsymbol{0}^{\mathrm{T}})}_{\boldsymbol{c}_{0i}^{\mathrm{T}}} \begin{pmatrix} \boldsymbol{x}_i(t) \\ \boldsymbol{x}_{\mathrm{r}i}(t) \end{pmatrix} \end{aligned}$$

and abbreviated as

$$\Sigma_{0i}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_{0i}(t) & = & \boldsymbol{A}_{0i}\boldsymbol{x}_{0i}(t) + \boldsymbol{b}_{0i}e_i(t), \quad \boldsymbol{x}_{0i}(0) = \bar{\boldsymbol{x}}_{i0} \\ y_i(t) & = & \boldsymbol{c}_{0i}^{\mathrm{T}}\boldsymbol{x}_{0i}(t). \end{array} \right. \tag{8}$$

Σ_{i0} is called the *extended agent*. The initial state is given by

$$\boldsymbol{x}_{0i}(0) = \begin{pmatrix} \boldsymbol{x}_i(0) \\ \boldsymbol{x}_{\mathrm{r}i}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{i0} \\ \boldsymbol{x}_{\mathrm{r}i0} \end{pmatrix}$$

and denoted by $\bar{\boldsymbol{x}}_{i0}$.

Assumption 1 *It is assumed that all extended agents are completely controllable by the input $e_i(t)$ and completely observable through the output $y_i(t)$.*

2.4 Overall system

The overall system consists of N extended agents (8) that are coupled by eqn. (6). The state of the overall system is given by the concatenation of the states of the extended agents:

$$\boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{x}_{01}(t) \\ \boldsymbol{x}_{02}(t) \\ \vdots \\ \boldsymbol{x}_{0N}(t) \end{pmatrix}.$$

The model of the overall system $\overline{\Sigma}$ is obtained by combining eqns. (6) and (8):

$$\text{Overall system } \overline{\Sigma}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) &=& \overline{\boldsymbol{A}}\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &=& \overline{\boldsymbol{C}}\boldsymbol{x}(t) \end{array} \right. \tag{9}$$

with

$$\begin{aligned} \overline{\boldsymbol{A}} &= \begin{pmatrix} \boldsymbol{A}_{01} & & & \\ & \boldsymbol{A}_{02} & & \\ & & \ddots & \\ & & & \boldsymbol{A}_{0N} \end{pmatrix} - \begin{pmatrix} \boldsymbol{b}_{01} & & & \\ & \boldsymbol{b}_{02} & & \\ & & \ddots & \\ & & & \boldsymbol{b}_{0N} \end{pmatrix} \boldsymbol{L} \begin{pmatrix} \boldsymbol{c}_{01}^{\mathrm{T}} & & & \\ & \boldsymbol{c}_{02}^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \boldsymbol{c}_{0N}^{\mathrm{T}} \end{pmatrix} \\ \overline{\boldsymbol{C}} &= \begin{pmatrix} \boldsymbol{c}_{01}^{\mathrm{T}} & & & \\ & \boldsymbol{c}_{02}^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \boldsymbol{c}_{0N}^{\mathrm{T}} \end{pmatrix}. \end{aligned} \tag{10}$$

3 Synchronisation condition

3.1 Representation of the synchronisation error

This section repeats a necessary and sufficient condition for the asymptotic synchronisation of a set of agents with individual dynamics given in [1] in Theorem 4.5.

Consider the extended agents Σ_{0i} described by eqn. (8) and assume that these agents have a nonempty intersection Σ_{s} described by eqn. (4). According to Theorem 4.4 in [1] there exist transformation matrices $\boldsymbol{T}_i$ such that the models of the extended agents can be decomposed into the virtual reference system and some remaining system:

$$\Sigma_{0i}: \left\{ \begin{array}{rcl} \dot{\tilde{\boldsymbol{x}}}_i(t) & = & \underbrace{\begin{pmatrix} \boldsymbol{A}_{\rm s} & \boldsymbol{A}_{{\rm q}i} \\ \boldsymbol{O} & \boldsymbol{A}_{{\rm p}i} \end{pmatrix}}_{\tilde{\boldsymbol{A}}_{0i}} \tilde{\boldsymbol{x}}_i(t) + \underbrace{\begin{pmatrix} \boldsymbol{b}_{{\rm q}i} \\ \boldsymbol{b}_{{\rm p}i} \end{pmatrix}}_{\tilde{\boldsymbol{b}}_{0i}} u_i(t), \quad \tilde{\boldsymbol{x}}_i(0) = \tilde{\boldsymbol{x}}_{i0} \\ y_i(t) & = & \underbrace{(\boldsymbol{c}_{\rm s}^{\rm T} \quad \boldsymbol{c}_{{\rm p}i}^{\rm T})}_{\tilde{\boldsymbol{c}}_{0i}^{\rm T}} \tilde{\boldsymbol{x}}_i(t), \qquad i = 1, 2, ..., N \end{array} \right. \tag{11}$$

with the state

$$\tilde{\boldsymbol{x}}_i(t) = \boldsymbol{T}_i^{-1} \boldsymbol{x}_{0i}(t) = \begin{pmatrix} \boldsymbol{x}_{{\rm q}i}(t) \\ \boldsymbol{x}_{{\rm p}i}(t) \end{pmatrix} \tag{12}$$

and with

$$\tilde{\boldsymbol{A}}_{0i} = \boldsymbol{T}_i^{-1} \boldsymbol{A}_{0i} \boldsymbol{T}_i, \quad \tilde{\boldsymbol{b}}_{0i} = \boldsymbol{T}_i^{-1} \boldsymbol{b}_{0i}, \quad \tilde{\boldsymbol{c}}_{0i}^{\rm T} = \boldsymbol{c}_{0i}^{\rm T} \boldsymbol{T}_i, \quad \tilde{\boldsymbol{x}}_{i0} = \boldsymbol{T}_i^{-1} \bar{\boldsymbol{x}}_{i0}$$

The required synchronisation condition is obtained by, first, considering the differences

$$e_{{\rm y}i}(t) = y_i(t) - y_1(t), \quad i = 2, 3, ..., N$$

between the agent outputs and finding a representation of the vector

$$\boldsymbol{e}_\Delta(t) = \left(e_{\rm y2}(t),\ e_{\rm y3}(t), ...,\ e_{{\rm y}N}(t)\right)^{\rm T} \tag{13}$$

in terms of the agent models and, second, by testing the stability of this representation.

The following lemma (Lemma 4.7 in [1]) gives a model for the synchronisation error (13).

Lemma 1 (Representation of the output differences)
Assume that the extended agents Σ_{0i}, $(i = 1, 2, ..., N)$ have the form (11) and possess a nonempty maximum intersection

$$\Sigma_{\rm s} = \left(\cap_{i=1}^N \Sigma_{0i}\right)^* = (\boldsymbol{A}_{\rm s}, \boldsymbol{c}_{\rm s}^{\rm T}).$$

Then the output differences (13) that appear in the overall system $\overline{\Sigma}$ described by eqn. (9) is represented by

$$\Sigma_\Delta: \left\{ \begin{array}{rcl} \dot{\breve{\boldsymbol{x}}}(t) & = & \breve{\boldsymbol{A}} \breve{\boldsymbol{x}}(t), \quad \breve{\boldsymbol{x}}(0) = \breve{\boldsymbol{x}}_0 \\ \boldsymbol{e}_\Delta(t) & = & \breve{\boldsymbol{C}} \breve{\boldsymbol{x}}(t) \end{array} \right. \tag{14}$$

with

$$\breve{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A}_{\rm p1} & & & \\ -\boldsymbol{T}_2\hat{\boldsymbol{A}}_{\rm q1} & \boldsymbol{A}_2 & & \\ \vdots & & \ddots & \\ -\boldsymbol{T}_N\hat{\boldsymbol{A}}_{\rm q1} & & & \boldsymbol{A}_N \end{pmatrix} - \begin{pmatrix} \boldsymbol{b}_{\rm p1} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ -\boldsymbol{T}_2\hat{\boldsymbol{b}}_{\rm q1} & \boldsymbol{b}_2 & & \\ \vdots & & \ddots & \\ -\boldsymbol{T}_N\hat{\boldsymbol{b}}_{\rm q1} & & & \boldsymbol{b}_N \end{pmatrix} \boldsymbol{L} \begin{pmatrix} \boldsymbol{c}_{\rm p1}^{\rm T} & & & \\ & \boldsymbol{c}_2^{\rm T} & & \\ & & \ddots & \\ & & & \boldsymbol{c}_N^{\rm T} \end{pmatrix}$$

$$\hat{\boldsymbol{b}}_{\rm q1} = \begin{pmatrix} \boldsymbol{b}_{\rm q1} \\ \boldsymbol{0} \end{pmatrix}, \quad \hat{\boldsymbol{A}}_{\rm q1} = \begin{pmatrix} \boldsymbol{A}_{\rm q1} \\ \boldsymbol{O} \end{pmatrix}, \quad \breve{\boldsymbol{C}} = \begin{pmatrix} -\boldsymbol{c}_{\rm p1}^{\rm T} & \boldsymbol{c}_2^{\rm T} & & \\ \vdots & & \ddots & \\ -\boldsymbol{c}_{\rm p1}^{\rm T} & & & \boldsymbol{c}_N^{\rm T} \end{pmatrix}. \tag{15}$$

The lemma is proved in [1], Appendix 4B.

3.2 Synchronisation criterion

As a by-product, the proof of Lemma 1 in [1] makes obvious that the system (14) may have any initial state $\breve{\boldsymbol{x}}_0$. Furthermore, as the extended agents are assumed to be completely observable, the pairs $(\boldsymbol{A}_{\rm p1}, \boldsymbol{c}_{\rm p1}^{\rm T})$ and $(\boldsymbol{A}_{0i}, \boldsymbol{c}_{0i}^{\rm T})$, $(i = 2, ..., N)$ are completely observable. As the matrix $\breve{\boldsymbol{A}}$ defined in eqn. (15) can be interpreted as the system matrix of an output feedback system and as output feedback cannot change the observability property of a system, the pair $(\breve{\boldsymbol{A}}, \breve{\boldsymbol{C}})$ is completely observable. Therefore, the synchronisation error $\boldsymbol{e}_\Delta(t)$ vanishes if and only if the matrix $\breve{\boldsymbol{A}}$ is Hurwitz.

This fact together with the Internal-Reference Principle (Theorem 4.4 in [1]) leads to the following synchronisation condition (proved as Theorem 4.5 in [1]).

Theorem 1 (Synchronisation condition)
A set of extended agents Σ_{0i} described by eqn. (8) is asymptotically synchronised by the networked controller (6) if and only if the following conditions are satisfied:

1. *There exists a nonempty intersection*

$$\Sigma_{\rm s} = \cap_{i=1}^{N} \Sigma_{0i} \neq \emptyset \tag{16}$$

   of all extended agents $\Sigma_{0i} = (\boldsymbol{A}_{0i}, \boldsymbol{b}_{0i}, \boldsymbol{c}_{0i}^{\rm T})$.

2. *The matrix $\breve{\boldsymbol{A}}$ defined in eqn. (15) is Hurwitz.*

3.3 Initial state for complete synchronisation

The proof of Lemma 1 also says for which initial states $\boldsymbol{x}_{0i}(0)$ of the extended agents all agent follow the same trajectory $y_{\mathrm{s}}(t)$ for all $t \geq 0$ as claimed in eqn. (1):

$$\check{\boldsymbol{x}}_0 = \mathbf{0}.$$

The transformation

$$\check{\boldsymbol{x}}_0 = \begin{pmatrix} \boldsymbol{x}_{\mathrm{p}10} \\ \boldsymbol{T}_2 \tilde{\boldsymbol{x}}_{20} \\ \vdots \\ \boldsymbol{T}_N \tilde{\boldsymbol{x}}_{N0} \end{pmatrix}$$

leads to the initial states

$$\begin{aligned} \boldsymbol{x}_{\mathrm{p}10} &= \mathbf{0} \\ \tilde{\boldsymbol{x}}_{i0} &= \mathbf{0}, \quad i = 2, 3, ..., N \end{aligned}$$

of the transformed extended agents (11) and to the initial states

$$\bar{\boldsymbol{x}}_{i0} = \bar{\boldsymbol{x}}_{i0}^{\mathrm{s}} \quad \text{with} \quad \bar{\boldsymbol{x}}_{i0}^{\mathrm{s}} = \boldsymbol{T}_i \begin{pmatrix} \boldsymbol{x}_{\mathrm{s}0} \\ \mathbf{0} \end{pmatrix} \quad \text{for some } \boldsymbol{x}_{\mathrm{s}0} \in \mathrm{I\!R}^{n_{\mathrm{s}}} \tag{17}$$

of the original extended agents (8). That is, all agents have to start in an initial state with the same q-component and vanishing p-component:

$$\begin{aligned} \boldsymbol{x}_{\mathrm{q}10} &= \boldsymbol{x}_{\mathrm{q}20} = ... = \boldsymbol{x}_{\mathrm{q}N0} = \boldsymbol{x}_{\mathrm{s}0} \\ \boldsymbol{x}_{\mathrm{p}10} &= \boldsymbol{x}_{\mathrm{p}20} = ... = \boldsymbol{x}_{\mathrm{p}N0} = \mathbf{0}. \end{aligned} \tag{18}$$

Then all agents are completely synchronised at the trajectory $y_{\mathrm{s}}(t)$ that is fixed by the q-component:

$$\begin{aligned} \dot{\boldsymbol{x}}_{\mathrm{s}}(t) &= \boldsymbol{A}_{\mathrm{s}} \boldsymbol{x}_{\mathrm{s}}(t), \quad \boldsymbol{x}_{\mathrm{s}}(0) = \boldsymbol{x}_{\mathrm{q}10} \\ y_{\mathrm{s}}(t) &= \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \boldsymbol{x}_{\mathrm{s}}(t). \end{aligned}$$

4 First method

4.1 Problem statement

Equation (17) gives the initial states $\bar{\boldsymbol{x}}_{i0}^{\mathrm{s}}$, $(i = 1, 2, ..., N)$ for which all agents follow the synchronous trajectory

$$y_{\mathrm{s}}(t) = \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \mathrm{e}^{\boldsymbol{A}_{\mathrm{s}} t} \boldsymbol{x}_{\mathrm{s}0}$$

and, hence, satisfy the requirement (1). This section considers the overall system for other initial states

$$\bar{\boldsymbol{x}}_{i0} \neq \bar{\boldsymbol{x}}_{i0}^{\mathrm{s}}$$

and shows how to determine the initial state $\boldsymbol{x}_{\mathrm{s}0}$ of the virtual reference system Σ_{s} to generate the synchronous trajectory $y_{\mathrm{s}}(t)$ to which all agents are asymptotically synchronised if the conditions of Theorem 1 are satisfied.

The following investigations are made for extended agent Σ_{0i} given in eqn. (8) under the following assumption:

Assumption 2 *The extended agents (8) have diagonalisable matrices $\boldsymbol{A}_{0i}$, $(i = 1, 2, ..., N)$.*

As a consequence, the virtual reference system has a diagonalisable matrix $\boldsymbol{A}_{\mathrm{s}}$.

4.2 Determination of the initial state of the virtual reference system as a linear combination of the initial agent states

Due to Assumption 2, there exist transformation matrices $\boldsymbol{T}_i$ that bring the agent models into the form

$$\Sigma_{0i}: \left\{ \begin{array}{rcl} \dot{\tilde{\boldsymbol{x}}}_i(t) & = & \tilde{\boldsymbol{A}}_{0i} \tilde{\boldsymbol{x}}_i(t) + \tilde{\boldsymbol{b}}_{0i} e_i(t), \quad \tilde{\boldsymbol{x}}_i(0) = \tilde{\boldsymbol{x}}_{i0} \\ y_i(t) & = & \tilde{\boldsymbol{c}}_{0i}^{\mathrm{T}} \tilde{\boldsymbol{x}}_i(t) \end{array} \right. \tag{19}$$

with the matrices

$$\tilde{\boldsymbol{A}}_{0i} = \boldsymbol{T}_i^{-1} \boldsymbol{A}_{0i} \boldsymbol{T}_i = \begin{pmatrix} \boldsymbol{A}_{\mathrm{s}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_{\mathrm{p}i} \end{pmatrix} \tag{20}$$

$$\tilde{\boldsymbol{b}}_{0i} = \boldsymbol{T}_i^{-1} \boldsymbol{b}_{0i} = \begin{pmatrix} \boldsymbol{b}_{\mathrm{q}i} \\ \boldsymbol{b}_{\mathrm{p}i} \end{pmatrix} \tag{21}$$

$$\tilde{\boldsymbol{c}}_{0i}^{\mathrm{T}} = \boldsymbol{c}_{0i}^{\mathrm{T}} \boldsymbol{T}_i = (\boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \;\; \boldsymbol{c}_{\mathrm{p}i}^{\mathrm{T}}) \tag{22}$$

and the state

$$\tilde{\boldsymbol{x}}_i = \begin{pmatrix} \boldsymbol{x}_{\mathrm{q}i}(t) \\ \boldsymbol{x}_{\mathrm{p}i}(t) \end{pmatrix} = \boldsymbol{T}_i^{-1} \bar{\boldsymbol{x}}_{0i}, \quad \tilde{\boldsymbol{x}}_{i0} = \begin{pmatrix} \boldsymbol{x}_{\mathrm{q}i0} \\ \boldsymbol{x}_{\mathrm{p}i0} \end{pmatrix} = \boldsymbol{T}_i^{-1} \bar{\boldsymbol{x}}_{i0}. \tag{23}$$

In contrast to eqn. (11), the matrices $\tilde{\boldsymbol{A}}_{0i}$ are block-diagonal. Therefore, the matrix $\breve{\boldsymbol{A}}$ to be considered in the synchronisation test of Theorem 1 simplifies to

$$\breve{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A}_{\mathrm{p}1} & & & \\ & \boldsymbol{A}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{A}_N \end{pmatrix} - \begin{pmatrix} \boldsymbol{b}_{\mathrm{p}1} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ -\boldsymbol{T}_2\hat{\boldsymbol{b}}_{\mathrm{q}1} & \boldsymbol{b}_2 & & \\ \vdots & & \ddots & \\ -\boldsymbol{T}_N\hat{\boldsymbol{b}}_{\mathrm{q}1} & & & \boldsymbol{b}_N \end{pmatrix} \boldsymbol{L} \begin{pmatrix} \boldsymbol{c}_{\mathrm{p}1}^{\mathrm{T}} & & & \\ & \boldsymbol{c}_2^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \boldsymbol{c}_N^{\mathrm{T}} \end{pmatrix} \tag{24}$$

with

$$\hat{\boldsymbol{b}}_{\mathrm{q}1} = \begin{pmatrix} \boldsymbol{b}_{\mathrm{q}1} \\ \boldsymbol{0} \end{pmatrix}.$$

The eigenvector matrix of $\boldsymbol{A}_{\mathrm{s}}$ is denoted by $\boldsymbol{V}_{\mathrm{s}}$. The diagonal matrix

$$\boldsymbol{T}_{\mathrm{b}i} = \begin{pmatrix} t_{i1} & & & \\ & t_{i2} & & \\ & & \ddots & \\ & & & t_{in_{\mathrm{s}}} \end{pmatrix}$$

is introduced that satisfies the relation

$$\boldsymbol{T}_{\mathrm{b}i}\boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{b}_{\mathrm{q}i} = \boldsymbol{1}, \quad i = 1, 2, ..., N. \tag{25}$$

Furthermore, the matrix

$$\boldsymbol{T}_{\mathrm{b}} = \sum_{i=1}^{N} \hat{w}_i \boldsymbol{T}_{\mathrm{b}i} \tag{26}$$

is defined.

The following theorem states the first main result of this report.

Theorem 2 (Initial state of the virtual reference system)
Consider extended agents (8) that satisfy Assumption 2 and assume that these extended agents together with the networked controller (6) satisfy the conditions of Theorem 1. Then the synchronous trajectory $y_{\mathrm{s}}(t)$ is generated by the virtual reference system (4) for the initial state

$$\boldsymbol{x}_{\mathrm{s}0} = \boldsymbol{V}_{\mathrm{s}}\boldsymbol{T}_{\mathrm{b}}^{-1} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{T}_{\mathrm{b}i}\boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{x}_{\mathrm{q}i0}, \tag{27}$$

where $\boldsymbol{w}^{\mathrm{T}} = (\hat{w}_1 \;\; \hat{w}_2 \; ... \; \hat{w}_N)$ is the normalised left eigenvector of the Laplacian matrix $\boldsymbol{L}$ belonging to the vanishing eigenvalue $\lambda_1\{\boldsymbol{L}\} = 0$. The diagonal matrices $\boldsymbol{T}_{\mathrm{b}i}$ are obtained from eqn. (25), the matrix $\boldsymbol{T}_{\mathrm{b}}$ from eqn. (26) and $\boldsymbol{x}_{\mathrm{q}i0}$, $(i = 1, 2, ..., N)$ are part of the initial states (23) of the transformed agents (19).

Proof. For the upper part of the state vector $\tilde{\boldsymbol{x}}_i(t)$, eqn. (19) yields

$$\dot{\boldsymbol{x}}_{\mathrm{q}i}(t) = \boldsymbol{A}_{\mathrm{s}}\boldsymbol{x}_{\mathrm{q}i}(t) + \boldsymbol{b}_{\mathrm{q}i}e_i(t), \quad \boldsymbol{x}_{\mathrm{q}i}(0) = \boldsymbol{x}_{\mathrm{q}i0}.$$

As the matrix $\boldsymbol{A}_{\mathrm{s}}$ is assumed to be diagonalisable, the state transformations

$$\hat{\boldsymbol{x}}_{\mathrm{q}i}(t) = \boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{x}_{\mathrm{q}i}(t), \quad i = 1, 2, ..., N$$

lead to the models

$$\dot{\hat{\boldsymbol{x}}}_{\mathrm{q}i}(t) = \hat{\boldsymbol{A}}_{\mathrm{s}}\hat{\boldsymbol{x}}_{\mathrm{q}i}(t) + \boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{b}_{\mathrm{q}i}e_i(t), \quad \hat{\boldsymbol{x}}_{\mathrm{q}i}(0) = \boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{x}_{\mathrm{q}i0}$$

$(i = 1, 2, ..., N)$ with

$$\hat{\boldsymbol{A}}_{\mathrm{s}} = \begin{pmatrix} \lambda_{\mathrm{s}1} & & & \\ & \lambda_{\mathrm{s}2} & & \\ & & \ddots & \\ & & & \lambda_{\mathrm{s}n_{\mathrm{s}}} \end{pmatrix}.$$

As the extended agents are assumed to be completely controllable, the pairs $(\boldsymbol{A}_{\mathrm{s}}, \boldsymbol{b}_{\mathrm{q}i})$, $(i = 1, 2, ..., N)$ and, hence, the pairs $(\hat{\boldsymbol{A}}_{\mathrm{s}}, \boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{b}_{\mathrm{q}i})$ are controllable. Consequently, the vectors $\boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{b}_{\mathrm{q}i}$ do not have any vanishing element and regular diagonal matrices $\boldsymbol{T}_{\mathrm{b}i}$, $(i = 1, 2, ..., N)$ exist which satisfy eqn. (25).

Now, the state

$$\hat{\boldsymbol{x}}_{\mathrm{s}}(t) = \boldsymbol{T}_{\mathrm{b}}^{-1}\sum_{i=1}^{N}\hat{w}_i\,\boldsymbol{T}_{\mathrm{b}i}\,\hat{\boldsymbol{x}}_{\mathrm{q}i}(t) \tag{28}$$

is introduced, for which the differential equation

$$\begin{aligned}
\dot{\hat{\boldsymbol{x}}}_{\mathrm{s}}(t) &= \boldsymbol{T}_{\mathrm{b}}^{-1}\sum_{i=1}^{N}\hat{w}_i\,\boldsymbol{T}_{\mathrm{b}i}\,\dot{\hat{\boldsymbol{x}}}_{\mathrm{q}i}(t) \\
&= \boldsymbol{T}_{\mathrm{b}}^{-1}\sum_{i=1}^{N}\hat{w}_i\,\boldsymbol{T}_{\mathrm{b}i}\hat{\boldsymbol{A}}_{\mathrm{s}}\hat{\boldsymbol{x}}_{\mathrm{q}i}(t) + \boldsymbol{T}_{\mathrm{b}}^{-1}\sum_{i=1}^{N}\hat{w}_i\,\underbrace{\boldsymbol{T}_{\mathrm{b}i}\boldsymbol{V}_{\mathrm{s}}^{-1}\boldsymbol{b}_{\mathrm{q}i}}_{\boldsymbol{1}}\sum_{j=1}^{N}(-l_{ij})y_j(t) \\
&= \hat{\boldsymbol{A}}_{\mathrm{s}}\boldsymbol{T}_{\mathrm{b}}^{-1}\sum_{i=1}^{N}\hat{w}_i\,\boldsymbol{T}_{\mathrm{b}i}\,\hat{\boldsymbol{x}}_{\mathrm{q}i}(t) - \boldsymbol{T}_{\mathrm{b}}^{-1}\sum_{j=1}^{N}\boldsymbol{1}\underbrace{\sum_{i=1}^{N}\hat{w}_i l_{ij}}_{=0}\,y_j(t) \\
&= \hat{\boldsymbol{A}}_{\mathrm{s}}\hat{\boldsymbol{x}}_{\mathrm{s}}(t)
\end{aligned}$$

is obtained, because

$$\hat{\boldsymbol{w}}^{\mathrm{T}}\boldsymbol{L} = \boldsymbol{0}^{\mathrm{T}}$$

holds. Note that $\hat{\boldsymbol{A}}_{\mathrm{s}}$, $\boldsymbol{T}_{\mathrm{b}}^{-1}$ and $\boldsymbol{T}_{\mathrm{b}i}$ are diagonal matrices and the synchronisation error $e_i(t)$ is given by eqn. (5). A back-transformation

$$\boldsymbol{x}_{\mathrm{s}}(t) = \boldsymbol{V}_{\mathrm{s}}\hat{\boldsymbol{x}}_{\mathrm{s}}(t)$$

yields eqn. (4) with the initial state (27).

The next step is to show that the trajectory generated by the model (4) is the synchronous trajectory to which all outputs $y_i(t)$ converge. The output equation

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{c}}_{01}^{\mathrm{T}} & & & \\ & \tilde{\boldsymbol{c}}_{02}^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \tilde{\boldsymbol{c}}_{0N}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{x}}_1(t) \\ \tilde{\boldsymbol{x}}_2(t) \\ \vdots \\ \tilde{\boldsymbol{x}}_N(t) \end{pmatrix}$$

of the overall system (9), which are represented in terms of the transformed states used in eqn. (19), can be re-written in terms of the state vectors

$$\hat{\boldsymbol{x}}_i(t) = \tilde{\boldsymbol{x}}_i(t) - \begin{pmatrix} \boldsymbol{x}_{\mathrm{q}1}(t) \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{\mathrm{q}i}(t) - \boldsymbol{x}_{\mathrm{q}1}(t) \\ \boldsymbol{x}_{\mathrm{p}i}(t) \end{pmatrix}, \quad i = 2, 3, ..., N$$

as

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}_{\mathrm{p}1}^{\mathrm{T}} & & & \\ & \tilde{\boldsymbol{c}}_{02}^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \tilde{\boldsymbol{c}}_{0N}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{\mathrm{p}1}(t) \\ \tilde{\boldsymbol{x}}_2(t) \\ \vdots \\ \tilde{\boldsymbol{x}}_N(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \\ \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \end{pmatrix} \boldsymbol{x}_{\mathrm{q}1}(t).$$

Under the conditions of Theorem 1 the model (14) is stable, which implies

$$\lim_{t\to\infty} \|\boldsymbol{x}_{\mathrm{p}1}(t)\| = 0$$

and

$$\lim_{t\to\infty} \|\tilde{\boldsymbol{x}}_i(t)\| = 0, \quad i = 2, 3, ..., N$$

and consequently

$$\lim_{t\to\infty} \|\boldsymbol{x}_{\mathrm{p}i}(t) - \boldsymbol{x}_{\mathrm{p}1}(t)\| = 0, \quad i = 2, 3, ..., N$$

and

$$\lim_{t\to\infty} \|y_i(t) - \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} \boldsymbol{x}_{\mathrm{q}1}(t)\| = 0 \quad i = 1, 2, ..., N.$$

It remains to show that

$$\lim_{t\to\infty} \|\boldsymbol{x}_{\mathrm{q}1}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\| = 0 \tag{29}$$

is valid. Equation (28) implies

$$\begin{aligned} \lim_{t\to\infty} \|\hat{\boldsymbol{x}}_{\mathrm{q}1}(t) - \hat{\boldsymbol{x}}_{\mathrm{s}}(t)\| &= \lim_{t\to\infty} \|\hat{\boldsymbol{x}}_{\mathrm{q}1}(t) - \boldsymbol{T}_{\mathrm{b}}^{-1} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{T}_{\mathrm{b}i} \, \hat{\boldsymbol{x}}_{\mathrm{q}i}(t)\| \\ &= \lim_{t\to\infty} \|\boldsymbol{T}_{\mathrm{b}}^{-1} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{T}_{\mathrm{b}i} \, \hat{\boldsymbol{x}}_{\mathrm{q}1}(t) - \boldsymbol{T}_{\mathrm{b}}^{-1} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{T}_{\mathrm{b}i} \, \hat{\boldsymbol{x}}_{\mathrm{q}i}(t)\| \\ &= \lim_{t\to\infty} \|\boldsymbol{T}_{\mathrm{b}}^{-1} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{T}_{\mathrm{b}i} \, (\hat{\boldsymbol{x}}_{\mathrm{q}1}(t) - \hat{\boldsymbol{x}}_{\mathrm{q}i}(t))\| \\ &= 0 \end{aligned}$$

and

$$\lim_{t\to\infty} \|\boldsymbol{x}_{\mathrm{q}1}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\| = \lim_{t\to\infty} \|\boldsymbol{V}_{\mathrm{s}}^{-1} (\hat{\boldsymbol{x}}_{\mathrm{q}1}(t) - \hat{\boldsymbol{x}}_{\mathrm{s}}(t))\| = 0$$

which proves the theorem. □

Algorithm 1 *Determine the initial state x_{s0} of the virtual reference system*

Given: Σ_{i0}, $(i = 1, 2, ..., N)$ of the form (8) satisfying Assumption 2

1. Determine the intersection $\Sigma_s = (\boldsymbol{A}_s, \boldsymbol{c}_s^T)$ of the extended agents. This step includes to find the transformation matrices $\boldsymbol{T}_i$, $(i = 1, 2, ..., N)$ and the decomposition of the extended agents as in eqns. (19) – (23).

2. Check the synchronisation condition: $\breve{\boldsymbol{A}}$ given in eqn. (24) has to be Hurwitz.

3. Determine the matrices $\boldsymbol{T}_{bi}$ and $\boldsymbol{T}_b$ according to eqns. (25) and (26) and the normalised left eigenvector $\hat{\boldsymbol{w}}^T$ of the Laplacian matrix $\boldsymbol{L}$.

4. Determine the required initial state (27) .

Result: Initial state x_{s0} of the virtual reference system.

In summary, the initial state x_{s0} can be determined by Algorithm 1.

4.3 Interpretation of the result

Equation (27) shows that the initial state $\boldsymbol{x}_{s0}$ of the virtual reference system is a linear combination of the initial states $\bar{\boldsymbol{x}}_{i0}$ of the extended agents. It generalises the proposition of eqn. (18) that $\boldsymbol{x}_{s0}$ only depends upon the q-component of the agent states.

To find a direct relation between $\boldsymbol{x}_{s0}$ and $\bar{\boldsymbol{x}}_{i0}$, $(i = 1, 2, ..., N)$, use eqn. (23) to get an expression for $\boldsymbol{x}_{qi0}$, which is a component of the transformed initial state $\boldsymbol{T}_i^{-1}\bar{\boldsymbol{x}}_{i0}$ of the extended agent Σ_{0i}. If the transformation matrix is decomposed as

$$\boldsymbol{T}_i^{-1} = \begin{pmatrix} \boldsymbol{T}_i^u \\ \boldsymbol{T}_i^l \end{pmatrix}$$

into its upper part $\boldsymbol{T}_i^u$ and its lower part $\boldsymbol{T}_i^l$, the initial state $\boldsymbol{x}_{qi0}$ used in eqn. (27) is represented as

$$\boldsymbol{x}_{qi0} = \boldsymbol{T}_i^u \bar{\boldsymbol{x}}_{i0}.$$

Then eqn. (27) reads as

$$\boldsymbol{x}_{s0} = \boldsymbol{V}_s \boldsymbol{T}_b^{-1} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{T}_{bi} \boldsymbol{V}_s^{-1} \boldsymbol{T}_i^u \bar{\boldsymbol{x}}_{i0}.$$

This representation refers to the weighted sum $\sum_{i=1}^{N} \hat{w}_i \boldsymbol{P}_i \bar{\boldsymbol{x}}_{i0}$ of the initial states $\bar{\boldsymbol{x}}_{i0}$, which are transformed by the matrix $\boldsymbol{P}_i = \boldsymbol{T}_{\mathrm{b}i} \boldsymbol{V}_{\mathrm{s}}^{-1} \boldsymbol{T}_i^{\mathrm{u}}$ given by the extended agents. This sum is finally transformed by $\boldsymbol{V}_{\mathrm{s}} \boldsymbol{T}_{\mathrm{b}}^{-1}$ to get $\boldsymbol{x}_{\mathrm{s}0}$:

$$\boldsymbol{x}_{\mathrm{s}0} = \boldsymbol{V}_{\mathrm{s}} \boldsymbol{T}_{\mathrm{b}}^{-1} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{P}_i \bar{\boldsymbol{x}}_{i0}. \tag{30}$$

Due to the individual properties of the agents, the initial states $\bar{\boldsymbol{x}}_{i0}$ of the extended agents have diverse effects on the initial state $\boldsymbol{x}_{\mathrm{s}0}$ of the virtual reference system. The influence of the communication structure is shown by the weightings $\hat{w}_i$, $(i = 1, 2, ..., N)$, which are the elements of the left eigenvector of the Laplacian matrix $\boldsymbol{L}$ for the vanishing eigenvalue.

If the transformed extended agents (19) have the same vectors $\boldsymbol{b}_{\mathrm{q}i} = \boldsymbol{b}_{\mathrm{q}}$, $(i = 1, 2, ..., N)$ the matrices $\boldsymbol{T}_{\mathrm{b}i}$ are the same, the relation $\boldsymbol{T}_{\mathrm{b}} = \boldsymbol{T}_{\mathrm{b}i}$ holds and these matrices disappear from eqn. (27):

$$\begin{aligned} \boldsymbol{x}_{\mathrm{s}0} &= \boldsymbol{V}_{\mathrm{s}} \sum_{i=1}^{N} \hat{w}_i \, \boldsymbol{V}_{\mathrm{s}}^{-1} \boldsymbol{x}_{\mathrm{q}i0} \\ &= \sum_{i=1}^{N} \hat{w}_i \boldsymbol{x}_{\mathrm{q}i0} \\ &= \sum_{i=1}^{N} \hat{w}_i \boldsymbol{T}_i^{\mathrm{u}} \bar{\boldsymbol{x}}_{i0}. \end{aligned}$$

$\boldsymbol{x}_{\mathrm{s}0}$ is the weighted sum of the components $\boldsymbol{x}_{\mathrm{q}i0}$ of the initial states $\bar{\boldsymbol{x}}_{i0}$ of the agents. This relation is similar to the corresponding relation (4.34) on p. 143 that is valid for identical agents:

$$\boldsymbol{x}_{\mathrm{s}0} = \sum_{i=1}^{N} \hat{w}_i \bar{\boldsymbol{x}}_{i0}.$$

4.4 Example: Synchronisation of three oscillators with different time constants

The following results extend Example 4.7 in [1].

Models. Three harmonic oscillators have first-order time lag elements with the time constants $T_1 = 1$, $T_2 = 2$ and $T_3 = 3$. The agent model (3) holds with

$$\boldsymbol{A}_i = \left(\begin{array}{cc:c} 0 & 3 & 1 \\ -3 & 0 & 1 \\ \hdashline 0 & 0 & -\frac{1}{T_i} \end{array}\right), \quad \boldsymbol{b}_i = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{T_i} \end{pmatrix}, \quad \boldsymbol{c}_i = \left(\begin{array}{cc:c} 1 & 10 & 0 \end{array}\right), \quad i = 1, 2, 3$$

and with the initial states

$$\boldsymbol{x}_{10} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_{20} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_{30} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The oscillators should be synchronised by a static networked controller, which implies that $\Sigma_{0i} = P_i$ holds. The structure of the matrix above reveals that Σ_{0i}, $(i = 1, 2, 3)$ have already the decomposed form (11) with

$$\boldsymbol{A}_{\mathrm{p}i} = -\frac{1}{T_i}, \quad \boldsymbol{b}_{\mathrm{p}i} = \frac{1}{T_i}, \quad \hat{\boldsymbol{A}}_{\mathrm{q}i} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{\boldsymbol{b}}_{\mathrm{q}i} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad i = 1, 2, 3$$

for the trivial transformation matrices $\boldsymbol{T}_i = \boldsymbol{I}$. A maximum intersection (16) has the state-space model (4) with the parameters

$$\boldsymbol{A}_{\mathrm{s}} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}, \quad \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} = (1 \;\; 10). \tag{31}$$

Hence, the agents satisfy the Internal-Reference Principle (16).

All-to-all couplings are described by the normalised Laplacian matrix

$$\hat{\boldsymbol{L}} = \begin{pmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{pmatrix} \tag{32}$$

which together with the gain $k = 0.1$ leads to the Laplacian matrix $\boldsymbol{L} = k\hat{\boldsymbol{L}}$ used in the networked controller

$$\boldsymbol{u}(t) = -\boldsymbol{L}\boldsymbol{y}(t).$$

The overall system is described by eqn. (9) with the matrices

$$\begin{aligned}
\overline{\boldsymbol{A}} &= \begin{pmatrix} \boldsymbol{A}_1 & & \\ & \boldsymbol{A}_2 & \\ & & \boldsymbol{A}_3 \end{pmatrix} - \begin{pmatrix} \boldsymbol{b}_1 & & \\ & \boldsymbol{b}_2 & \\ & & \boldsymbol{b}_3 \end{pmatrix} \boldsymbol{L} \begin{pmatrix} \boldsymbol{c}_1^{\mathrm{T}} & & \\ & \boldsymbol{c}_2^{\mathrm{T}} & \\ & & \boldsymbol{c}_3^{\mathrm{T}} \end{pmatrix} \\
&= \left(\begin{array}{ccc:ccc:ccc}
0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.1 & -1 & -1 & 0.05 & 0.5 & 0 & 0.05 & 0.5 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 1 & 0 & 0 & 0 \\
0.025 & 0.25 & 0 & -0.05 & -0.5 & -0.5 & 0.025 & 0.25 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 1 \\
0.0167 & 0.1667 & 0 & 0.0167 & 0.1667 & 0 & -0.033 & -0.333 & -0.333
\end{array}\right) \\
\overline{\boldsymbol{C}} &= \left(\begin{array}{ccc:ccc:ccc}
1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 0
\end{array}\right).
\end{aligned}$$

The matrix $\overline{\boldsymbol{A}}$ has, among others, the eigenvalues $\pm \mathrm{j}\,3$, which originate from the common dynamics.

Synchronous trajectory. To determine the initial state $\boldsymbol{x}_{\mathrm{s}0}$ of the virtual reference system (4) with the parameters (31) according to Algorithm 1 the oscillators have to be transformed into the form (19) with the matrices (20) – (22). The matrix

$$\boldsymbol{A}_i = \left(\begin{array}{cc:c} 0 & \omega & 1 \\ -\omega & 0 & 1 \\ \hdashline 0 & 0 & -\frac{1}{T_i} \end{array} \right)$$

is transformed with the matrix

$$\boldsymbol{T}_i = \left(\begin{array}{cc:c} 1 & 0 & s_i \\ 0 & 1 & t_i \\ \hdashline 0 & 0 & 1 \end{array} \right) \quad \text{with} \quad s_i = \frac{a_{ii} + \omega}{a_{ii}^2 + \omega^2}, \quad t_i = \frac{a_{ii} - \omega}{a_{ii}^2 + \omega^2}, \quad a_{ii} = -\frac{1}{T_i}$$

to get

$$\boldsymbol{T}_i^{-1} \boldsymbol{A}_i \boldsymbol{T}_i = \left(\begin{array}{cc:c} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ \hdashline 0 & 0 & -\frac{1}{T_i} \end{array} \right), \quad \boldsymbol{T}_i^{-1} \boldsymbol{b}_i = \begin{pmatrix} -\frac{s_i}{T_i} \\ -\frac{t_i}{T_i} \\ \frac{1}{T_i} \end{pmatrix}, \quad \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{T}_i = (1 \;\; 10 \;\; s_i{+}10t_i)$$

all of which have the form required in eqns. (20) and (21). Equation (23) leads to the transformed initial state

$$\boldsymbol{x}_{\mathrm{q}i0} = \begin{pmatrix} x_{i01} - s_i x_{i03} \\ x_{i02} - t_i x_{i03} \end{pmatrix} \quad \text{for} \quad \boldsymbol{x}_{i0} = (x_{i01},\; x_{i02},\; x_{i03})^{\mathrm{T}}.$$

For $\omega = 3$ one gets the following model elements for the first oscillator and similar numerical results for the other two oscillators:

$$\begin{aligned} \boldsymbol{T}_1 &= \begin{pmatrix} 1 & 0 & -0.0642 \\ 0 & 1 & -0.1193 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{b}_{\mathrm{q}1} = \begin{pmatrix} 0.6422 \\ 1.1927 \end{pmatrix}, \quad \hat{\boldsymbol{b}}_{\mathrm{q}1} = \begin{pmatrix} 0.6422 \\ 1.1927 \\ 0 \end{pmatrix} \\ \boldsymbol{A}_{\mathrm{p}1} &= -10, \quad \boldsymbol{b}_{\mathrm{p}1} = 10, \quad \boldsymbol{c}_{\mathrm{p}1}^{\mathrm{T}} = 0. \end{aligned}$$

The matrix $\breve{\boldsymbol{A}}$ given in eqn. (24)

$$\breve{\boldsymbol{A}} = \left(\begin{array}{c:ccc:ccc} -8.7431 & 0.5 & 5 & 0 & 0.5 & 5 & 0 \\ \hdashline -0.0807 & -0.0321 & 2.6789 & 1 & -0.0321 & -0.3211 & 0 \\ -0.1499 & -3.0596 & -0.5963 & 1 & -0.0596 & -0.5963 & 0 \\ -0.3142 & -0.5 & -5 & -5 & 0.2500 & 2.5 & 0 \\ \hdashline -0.0807 & -0.0321 & -0.3211 & 0 & -0.0321 & 2.6789 & 1 \\ -0.1499 & -0.0596 & -0.5963 & 0 & -3.0596 & -0.5963 & 1 \\ -0.0209 & 0.0167 & 0.1667 & 0 & -0.0333 & -0.3333 & -0.3333 \end{array} \right)$$

is Hurwitz, which means that the three oscillators are synchronised by the controller given.

The virtual reference system gets its canonical normal form by applying the transformation matrix

$$\boldsymbol{V}_{\rm s} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ {\rm j} & -{\rm j} \end{pmatrix} \quad \text{with} \quad \boldsymbol{V}_{\rm s}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -{\rm j} \\ 1 & {\rm j} \end{pmatrix},$$

with the results

$$\boldsymbol{V}_{\rm s}^{-1}\boldsymbol{b}_{{\rm q}i} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -{\rm j} \\ 1 & {\rm j} \end{pmatrix} \begin{pmatrix} -\frac{s_i}{T_i} \\ -\frac{t_i}{T_i} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{s_i}{T_i} + {\rm j}\frac{t_i}{T_i} \\ -\frac{s_i}{T_i} - {\rm j}\frac{t_i}{T_i} \end{pmatrix}$$

and

$$\boldsymbol{T}_{{\rm b}i} = \begin{pmatrix} \frac{\sqrt{2}T_i}{-s_i+{\rm j}t_i} & 0 \\ 0 & \frac{\sqrt{2}T_i}{-s_i-{\rm j}t_i} \end{pmatrix}.$$

With the feedback gains chosen above, the Laplacian matrix is

$$\boldsymbol{L} = \begin{pmatrix} 0.1 & -0.05 & -0.05 \\ -0.05 & 0.1 & -0.05 \\ -0.05 & -0.05 & 0.1 \end{pmatrix}$$

with the normalised left eigenvector

$$\boldsymbol{w}^{\rm T} = \boldsymbol{1}^{\rm T}(\boldsymbol{L} - \boldsymbol{1}\boldsymbol{1}^{\rm T})^{-1} = (0.333,\ 0.333,\ 0.333).$$

With these results the following model elements for the first oscillator are obtained:

$$\begin{aligned} \boldsymbol{V}_{\rm s} &= \begin{pmatrix} 0.7071 & 0.7071 \\ {\rm j}0.7071 & -{\rm j}0.7071 \end{pmatrix}, \quad \boldsymbol{V}_{\rm s}^{-1}\boldsymbol{b}_{\rm q1} = \begin{pmatrix} 0.4541 - {\rm j}0.8433 \\ 0.4541 + {\rm j}0.8433 \end{pmatrix} \\ \boldsymbol{T}_{\rm b1} &= \begin{pmatrix} 0.4950 + {\rm j}0.9192 & 0 \\ 0 & 0.4950 - {\rm j}0.9192 \end{pmatrix} \\ \boldsymbol{T}_{\rm b} &= \begin{pmatrix} -1.6263 + {\rm j}3.0406 & 0 \\ 0 & -1.6263 - {\rm j}3.0406 \end{pmatrix}. \end{aligned}$$

The initial states of the extended agents are obtained from the state transformation (23):

$$\boldsymbol{x}_{\rm q10} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_{\rm q20} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_{\rm q30} = \begin{pmatrix} -0.2927 \\ 0.3659 \end{pmatrix}.$$

Finally, eqn. (27) leads to the required result:

$$\boldsymbol{x}_{\mathrm{s}0} = \begin{pmatrix} -0.3255 \\ 0.4926 \end{pmatrix}.$$

$\boldsymbol{x}_{\mathrm{s}0}$ can be stated as the linear combination (30) of the initial states of the extended agents as

$$\begin{aligned}
\boldsymbol{x}_{\mathrm{s}0} &= \begin{pmatrix} -0.0967 - \mathrm{j}0.1808 & -0.0967 + \mathrm{j}0.1808 \\ 0.1808 - \mathrm{j}0.0967 & 0.1808 + \mathrm{j}0.0967 \end{pmatrix} \cdot \\
&\quad \Big[0.3333 \begin{pmatrix} 0.3500 + \mathrm{j}0.6500 & 0.6500 - \mathrm{j}0.3500 & 0.1 \\ 0.3500 - \mathrm{j}0.6500 & 0.6500 + \mathrm{j}0.3500 & 0.1 \end{pmatrix} \boldsymbol{x}_{10} \\
&\quad + 0.3333 \begin{pmatrix} 0.3500 + \mathrm{j}0.6500 & 0.6500 - \mathrm{j}0.3500 & 0.1735 - \mathrm{j}0.0441 \\ 0.3500 - \mathrm{j}0.6500 & 0.6500 + \mathrm{j}0.3500 & 0.1735 + \mathrm{j}0.0441 \end{pmatrix} \boldsymbol{x}_{20} \\
&\quad + 0.3333 \begin{pmatrix} 0.3500 + \mathrm{j}0.6500 & 0.6500 - \mathrm{j}0.3500 & 0.1354 - \mathrm{j}0.3183 \\ 0.3500 - \mathrm{j}0.6500 & 0.6500 + \mathrm{j}0.3500 & 0.1354 + \mathrm{j}0.3183 \end{pmatrix} \boldsymbol{x}_{30} \Big] \\
&= \begin{pmatrix} -0.3255 \\ 0.4926 \end{pmatrix}.
\end{aligned}$$

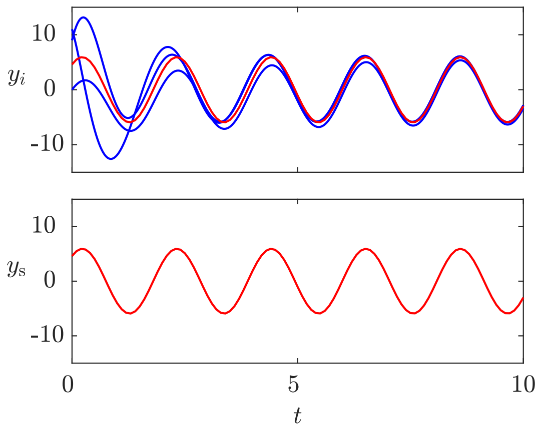

Fig. 2: Synchronous trajectory (below) and synchronised behaviour of three oscillators (above)

Figure 2 shows how the oscillators synchronise when starting in different initial states. The lower part depicts the synchronous trajectory that is generated by the virtual reference system for the initial state $\boldsymbol{x}_{\mathrm{s}0}$ given above.

5 Second method

5.1 Problem statement

The preceding section has elaborated the representation (27) of the initial state $\boldsymbol{x}_{\mathrm{s}0}$ of the virtual reference system in terms of the initial states of the agents. The result shows a direct relation between $\boldsymbol{x}_{\mathrm{s}0}$, properties of the Laplacian matrix $\boldsymbol{L}$ of the communication network and properties of the extended agents. In particular, it was shown that $\boldsymbol{x}_{\mathrm{s}0}$ is a linear combination of $\bar{\boldsymbol{x}}_{i0}$, $(i = 1, 2, ..., N)$.

However, these results use Assumption 2, which simplifies the representation of the $\boldsymbol{x}_{\mathrm{q}}$-component of the agent states (12) due to the block-diagonal matrix $\tilde{\boldsymbol{A}}_{0i}$ in eqn. (20). Non-diagonalisable matrices occur, in particular, in the synchronisation of vehicles or robots that are represented by integrators or double-integrators. This section derives explicit relations for $\boldsymbol{x}_{\mathrm{s}0}$ in dependence upon $\bar{\boldsymbol{x}}_{i0}$ for such systems. The next section demonstrates the main idea by investigating two coupled agents. Later on, the results are extended to overall systems with N agents.

5.2 Analysis of two coupled agents with integrator dynamics

Consider the extended agents

$$\Sigma_{01}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_1(t) & = & \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{1}{T_1} & \frac{1}{T_1} \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x}_1(t) + \begin{pmatrix} 0 \\ \frac{a}{T_1} \\ 1 \end{pmatrix} e_1(t), \quad \boldsymbol{x}_1(0) = \boldsymbol{x}_{10} \\ y_1(t) & = & (1 \;\; 0 \;\; 0)\, \boldsymbol{x}_1(t) \end{array} \right. \tag{33}$$

and

$$\Sigma_{02}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_2(t) & = & \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{T_2} \end{pmatrix} \boldsymbol{x}_2(t) + \begin{pmatrix} 0 \\ \frac{1}{T_2} \end{pmatrix} e_2(t), \quad \boldsymbol{x}_2(0) = \boldsymbol{x}_{20} \\ y_2(t) & = & (1 \;\; 0)\, \boldsymbol{x}_2(t). \end{array} \right. \tag{34}$$

Both extended agents include an integrator together with a first-order dynamics as an agent P_i to be synchronised. Σ_{01} results from the extension of this agent by a second integrator, whereas $\Sigma_{02} = P_2$ holds.

The first integrator in eqns. (33) and (34) represents the common dynamics of Σ_{01} and Σ_{02}, which is represented by a first-order virtual reference system (4) with the scalar parameters

$$\boldsymbol{A}_{\mathrm{s}} = 0 \quad \text{and} \quad \boldsymbol{c}_{\mathrm{s}}^{\mathrm{T}} = 1.$$

Σ_{01} does not satisfy Assumption 2. Both extended agents have the structure required in eqn. (11). For $t \to \infty$ the integrator agents reach the consensus state $x_{\mathrm{s}0}$, which should be determined.

Overall system. With the networked controller represented by

$$\begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = -k\boldsymbol{L} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \quad \text{with} \quad \boldsymbol{L} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

the overall system has the model (9) with the matrices

$$\begin{aligned}
\overline{\boldsymbol{A}} &= \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{T_1} & \frac{1}{T_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{T_2} \end{array}\right) \\
&\quad - \left(\begin{array}{cc} 0 & 0 \\ \frac{a}{T_1} & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & \frac{1}{T_2} \end{array}\right) k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \\
&= \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ -\frac{ak}{T_1} & -\frac{1}{T_1} & \frac{1}{T_1} & \frac{ak}{T_1} & 0 \\ -k & 0 & 0 & k & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \frac{k}{T_2} & 0 & 0 & -\frac{k}{T_2} & -\frac{1}{T_2} \end{array}\right) \\
\overline{\boldsymbol{C}} &= \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).
\end{aligned}$$

With the notation of the agent states of eqn. (12) the model is structured as follows:

$$\overline{\Sigma} : \begin{cases}
\left(\begin{array}{c} \dot{x}_{\mathrm{q}1}(t) \\ \hline \dot{x}_{\mathrm{p}11}(t) \\ \dot{x}_{\mathrm{p}12}(t) \\ \dot{x}_{\mathrm{q}2}(t) \\ \dot{x}_{\mathrm{p}2}(t) \end{array}\right) = \left(\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 0 \\ \hline -\frac{ak}{T_1} & -\frac{1}{T_1} & \frac{1}{T_1} & \frac{ak}{T_1} & 0 \\ -k & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{k}{T_2} & 0 & 0 & -\frac{k}{T_2} & -\frac{1}{T_2} \end{array}\right) \underbrace{\left(\begin{array}{c} x_{\mathrm{q}1}(t) \\ \hline x_{\mathrm{p}11}(t) \\ x_{\mathrm{p}12}(t) \\ x_{\mathrm{q}2}(t) \\ x_{\mathrm{p}2}(t) \end{array}\right)}_{\boldsymbol{x}(t)} \\[2ex]
\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \cdot \left(\begin{array}{c} x_{\mathrm{q}1}(t) \\ \hline x_{\mathrm{p}11}(t) \\ x_{\mathrm{p}12}(t) \\ x_{\mathrm{q}2}(t) \\ x_{\mathrm{p}2}(t) \end{array}\right).
\end{cases}$$

Consequently, the initial states $\boldsymbol{x}_{10}$ and $\boldsymbol{x}_{20}$ are composed of the following elements:

$$\boldsymbol{x}_{10} = \begin{pmatrix} x_{\mathrm{q}10} \\ x_{\mathrm{p}110} \\ x_{\mathrm{p}120} \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_{20} = \begin{pmatrix} x_{\mathrm{q}20} \\ x_{\mathrm{p}20} \end{pmatrix}.$$

Analysis of the overall system. The idea of the following analysis, which should end up with the required representation of the initial state x_{s0} of the virtual reference system, is to isolate the first component of the state vector as the state variable of Σ_s.

The state transformation

$$\hat{\boldsymbol{x}}(t) = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\hat{\boldsymbol{T}}^{-1}} \boldsymbol{x}(t) \quad \text{with} \quad \hat{\boldsymbol{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

introduces as the new fourth component the difference $\hat{x}_4(t) = x_{q2}(t) - x_{q1}(t)$ and leads to the representation of the overall system by

$$\overline{\Sigma} : \begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \left(\begin{array}{c:cccc} 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & -\frac{1}{T_1} & \frac{1}{T_1} & \frac{ak}{T_1} & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{k}{T_2} & -\frac{1}{T_2} \end{array}\right) \hat{\boldsymbol{x}}(t) \\ \boldsymbol{y}(t) = \left(\begin{array}{c:cccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right) \hat{\boldsymbol{x}}(t) \end{cases}$$

with a zero column in the matrix and the initial state

$$\hat{\boldsymbol{x}}(0) = \begin{pmatrix} x_{q10} \\ \hdashline x_{p110} \\ x_{p120} \\ x_{q20} - x_{q10} \\ x_{p20} \end{pmatrix}.$$

The matrix is abbreviated as

$$\hat{\boldsymbol{A}} = \hat{\boldsymbol{T}}^{-1} \overline{\boldsymbol{A}} \hat{\boldsymbol{T}} = \begin{pmatrix} 0 & \boldsymbol{a}_{12}^{\mathrm{T}} \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{22} \end{pmatrix} \quad \text{with} \quad \hat{\boldsymbol{A}}_{22} = \begin{pmatrix} -\frac{1}{T_1} & \frac{1}{T_1} & \frac{ak}{T_1} & 0 \\ 0 & 0 & k & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{T_2} & -\frac{1}{T_2} \end{pmatrix}$$

$$\text{and} \quad \boldsymbol{a}_{12}^{\mathrm{T}} = (1\ 0\ 0\ 0).$$

The matrix $\boldsymbol{A}_{22}$ has to be Hurwitz for the overall system to be synchronised. For $T_1 = T_2 = T$ and

$$a > T - 1$$

this condition is satisfied, because for these parameters the characteristic polynomial can be decomposed as

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}_{22}) = \left(\lambda + \frac{1}{T}\right)\left(\lambda^3 + \frac{1}{T}\lambda^2 + \frac{k}{T}(a+1)\lambda + \frac{k}{T}\right)$$

and the Routh-Hurwitz criterion leads to the inequalities above.

The next step uses another state transformation

$$\check{\boldsymbol{x}} = \underbrace{\begin{pmatrix} 1 & -\boldsymbol{a}_{12}^{\mathrm{T}}\hat{\boldsymbol{A}}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}}_{\check{\boldsymbol{T}}^{-1}} \hat{\boldsymbol{x}}(t) \quad \text{with} \qquad \check{\boldsymbol{T}} = \begin{pmatrix} 1 & \boldsymbol{a}_{12}^{\mathrm{T}}\hat{\boldsymbol{A}}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}$$

$$\boldsymbol{a}_{12}^{\mathrm{T}}\hat{\boldsymbol{A}}_{22}^{-1} = (0\ -1\ -1\ -T_2)$$

to give the matrix a zero row in the first line

$$\overline{\Sigma} : \begin{cases} \dot{\check{\boldsymbol{x}}}(t) = \left(\begin{array}{c:cccc} 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & -\frac{1}{T_1} & \frac{1}{T_1} & \frac{ak}{T_1} & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{k}{T_2} & -\frac{1}{T_2} \end{array}\right) \check{\boldsymbol{x}}(t) \\ \boldsymbol{y}(t) = \left(\begin{array}{c:cccc} 1 & 0 & -1 & -1 & -T_2 \\ 1 & 0 & -1 & 0 & -T_2 \end{array}\right) \check{\boldsymbol{x}}(t). \end{cases}$$

The initial state

$$\check{\boldsymbol{x}}(0) = \begin{pmatrix} 1 & \boldsymbol{a}_{12}^{\mathrm{T}}\hat{\boldsymbol{A}}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \hat{\boldsymbol{x}}(0)$$

has the first component

$$\begin{aligned} \check{x}_1(0) &= x_{\mathrm{q}10} + x_{\mathrm{p}120} + (x_{\mathrm{q}20} - x_{\mathrm{q}10}) + T_2 x_{\mathrm{p}20} \\ &= x_{\mathrm{p}120} + x_{\mathrm{q}20} + T_2 x_{\mathrm{p}20}. \end{aligned} \tag{35}$$

With the decomposition of the state vector as

$$\check{\boldsymbol{x}}(t) = \begin{pmatrix} x_{\mathrm{q}1}(t) \\ \check{\boldsymbol{x}}_2(t) \end{pmatrix}$$

the model implies

$$\lim_{t\to\infty} \|\check{\boldsymbol{x}}_2(t)\| = 0$$
$$\lim_{t\to\infty} y_1(t) = \check{x}_1(0).$$

Hence, the first component (35) of the initial state of the last model is the required initial state $x_{\mathrm{s}0}$ of the first-order virtual reference system:

$$x_{\mathrm{s}0} = x_{13}(0) + x_{21}(0) + T_2 x_{22}(0). \tag{36}$$

Figure 3 shows the behaviour of the two agents with the following parameters

$$T_1 = 1, \quad T_2 = 0.5, \quad a = 2, \quad k = 0.2$$

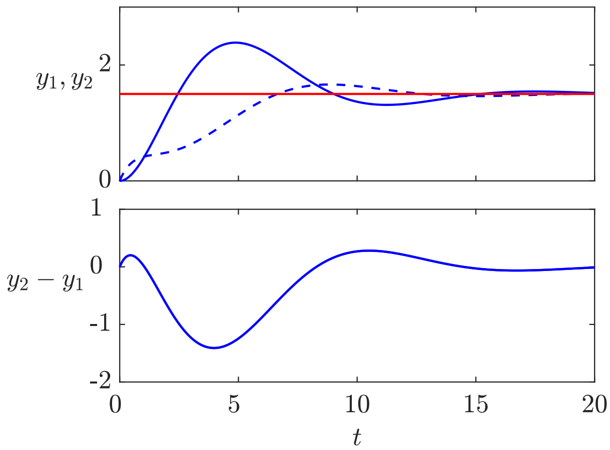

Fig. 3: Behaviour of two agents with integrator dynamics

and the initial states

$$\boldsymbol{x}_{10} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_{20} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The consensus value is obtained from eqn. (36):

$$x_{s0} = 1 + 0 + 0.5 \cdot 1 = 1.5.$$

5.3 Extension of the method for N agents

A generalisation of the result of the preceding section is obtained by using the state transformations and the results of the proof of Lemma 4.7 on pp. 263–265 in [1]. The extended agents have models of the form (33) and (34) both of which can be generalised as

$$\Sigma_{0i}: \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_i(t) & = & \underbrace{\left(\begin{array}{c:c} 0 & \boldsymbol{a}_{qi}^{\mathrm{T}} \\ \hdashline \boldsymbol{0} & \boldsymbol{A}_{pi} \end{array}\right)}_{\boldsymbol{A}_i} \boldsymbol{x}_i(t) + \underbrace{\left(\begin{array}{c} 0 \\ \hdashline \boldsymbol{b}_{pi} \end{array}\right)}_{\boldsymbol{b}_i} e_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ y_i(t) & = & \underbrace{\left(\begin{array}{c:c} 1 & \boldsymbol{0}^{\mathrm{T}} \end{array}\right)}_{\boldsymbol{c}_i^{\mathrm{T}}} \boldsymbol{x}_i(t) \end{array} \right. \tag{37}$$

with

$$\boldsymbol{a}_{qi}^{\mathrm{T}} = (1 \;\; \boldsymbol{0}^{\mathrm{T}})$$

and the state vector

$$\boldsymbol{x}_i(t) = \begin{pmatrix} x_{qi}(t) \\ \boldsymbol{x}_{pi}(t) \end{pmatrix}.$$

The state transformation (4.250)

$$\hat{\boldsymbol{x}}_i(t) = \boldsymbol{x}_i(t) - \begin{pmatrix} x_{\mathrm{q}1}(t) \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} x_{\mathrm{q}i}(t) - x_{\mathrm{q}1}(t) \\ \boldsymbol{x}_{\mathrm{p}i}(t) \end{pmatrix}, \quad i = 2, 3, ..., N \tag{38}$$

introduces the new state vectors $\hat{\boldsymbol{x}}_i(t)$, for which, after a lengthy calculation, the model (4.254) is obtained

$$\dot{\hat{\boldsymbol{x}}}(t) = \hat{\boldsymbol{A}}\hat{\boldsymbol{x}}(t)$$

with

$$\hat{\boldsymbol{x}}(t) = \begin{pmatrix} \boldsymbol{x}_{\mathrm{p}1}(t) \\ \hat{\boldsymbol{x}}_2(t) \\ \vdots \\ \hat{\boldsymbol{x}}_N(t) \end{pmatrix} \tag{39}$$

$$\hat{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A}_{\mathrm{p}1} & & & \\ -\hat{\boldsymbol{A}}_{\mathrm{q}1} & \boldsymbol{A}_2 & & \\ \vdots & & \ddots & \\ -\hat{\boldsymbol{A}}_{\mathrm{q}1} & & & \boldsymbol{A}_N \end{pmatrix} - \begin{pmatrix} \boldsymbol{b}_{\mathrm{p}1} & & & \\ & \boldsymbol{b}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{b}_N \end{pmatrix} \boldsymbol{L} \begin{pmatrix} \mathbf{0}^{\mathrm{T}} & & & \\ & \tilde{\boldsymbol{c}}_2^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \tilde{\boldsymbol{c}}_N^{\mathrm{T}} \end{pmatrix} \tag{40}$$

and

$$\hat{\boldsymbol{A}}_{\mathrm{q}1} = \begin{pmatrix} \boldsymbol{a}_{\mathrm{q}1}^{\mathrm{T}} \\ \boldsymbol{O} \end{pmatrix}.$$

In the matrix $\hat{\boldsymbol{A}}_{\mathrm{q}1}$ that appears in $\hat{\boldsymbol{A}}$ several times, the number of zero rows depends upon the dynamical order of the i–th agent. The matrix $\hat{\boldsymbol{A}}$ is Hurwitz if and only if the synchronisation condition of Theorem 1 is satisfied.

To get the agent output $y_1(t)$, the state variable $x_{\mathrm{q}1}(t)$ has to be added, which follows the differential equation

$$\dot{x}_{\mathrm{q}1}(t) = \boldsymbol{a}_{\mathrm{q}1}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{p}1}(t)$$

obtained from eqn. (37). The extended model is

$$\overline{\Sigma}: \begin{cases} \begin{pmatrix} \dot{x}_{\mathrm{q}1}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\boldsymbol{a}}_{\mathrm{q}1}^{\mathrm{T}} \\ \mathbf{0} & \hat{\boldsymbol{A}} \end{pmatrix} \begin{pmatrix} x_{\mathrm{q}1}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} \\ y_1(t) = x_{\mathrm{q}1}(t) \end{cases}$$

with

$$\tilde{\boldsymbol{a}}_{\mathrm{q}1}^{\mathrm{T}} = (\boldsymbol{a}_{\mathrm{q}1}^{\mathrm{T}} \;\; \mathbf{0}^{\mathrm{T}}).$$

The transformation

$$\breve{\boldsymbol{x}}(t) = \underbrace{\begin{pmatrix} 1 & -\tilde{\boldsymbol{a}}_{\mathrm{q}1}^{\mathrm{T}}\hat{\boldsymbol{A}}^{-1} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}}_{\breve{\boldsymbol{T}}^{-1}} \begin{pmatrix} x_{\mathrm{q}1}(t) \\ \hat{\boldsymbol{x}}(t) \end{pmatrix} \quad \text{with} \quad \breve{\boldsymbol{T}} = \begin{pmatrix} 1 & \tilde{\boldsymbol{a}}_{\mathrm{q}1}^{\mathrm{T}}\hat{\boldsymbol{A}}^{-1} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}$$

yields the new representation

$$\overline{\Sigma}: \left\{ \begin{array}{rcl} \dot{\check{\boldsymbol{x}}}(t) & = & \begin{pmatrix} 0 & \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{0} & \hat{\boldsymbol{A}} \end{pmatrix} \check{\boldsymbol{x}}(t) \\ y_1(t) & = & (1 \;\; \tilde{\boldsymbol{a}}_{\mathrm{q1}}^{\mathrm{T}} \hat{\boldsymbol{A}}^{-1}) \, \check{\boldsymbol{x}}(t) \end{array} \right.$$

with the initial state

$$\check{\boldsymbol{x}}(0) = \begin{pmatrix} 1 & -\tilde{\boldsymbol{a}}_{\mathrm{q1}}^{\mathrm{T}} \hat{\boldsymbol{A}}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} x_{\mathrm{q1}}(0) \\ \hat{\boldsymbol{x}}(0) \end{pmatrix}. \tag{41}$$

Since $\hat{\boldsymbol{A}}$ is Hurwitz, the relation

$$\lim_{t \to \infty} y_1(t) = \check{x}_{10}$$

holds, which means that $x_{\mathrm{s0}} = \check{x}_{10}$ is the required result. To find $\check{x}_{10}$ in a better representation, eqn. (38) is used to get

$$\hat{\boldsymbol{x}}_i(0) = \boldsymbol{x}_{i0} - \begin{pmatrix} x_{\mathrm{q1}}(0) \\ \boldsymbol{0} \end{pmatrix}.$$

Hence, from eqn. (39) the relation

$$\hat{\boldsymbol{x}}(0) = \begin{pmatrix} \boldsymbol{x}_{\mathrm{p1}}(0) \\ \boldsymbol{x}_{20} - \begin{pmatrix} x_{\mathrm{q1}}(0) \\ \boldsymbol{0} \end{pmatrix} \\ \vdots \\ \boldsymbol{x}_{N0} - \begin{pmatrix} x_{\mathrm{q1}}(0) \\ \boldsymbol{0} \end{pmatrix} \end{pmatrix}$$

and from eqn. (41) the final result

$$\boxed{x_{s0} = (1 \quad - \tilde{\boldsymbol{a}}_{\mathrm{q1}}^{\mathrm{T}} \hat{\boldsymbol{A}}^{-1}) \begin{pmatrix} \boldsymbol{x}_{10} \\ \boldsymbol{x}_{20} - \begin{pmatrix} x_{\mathrm{q1}}(0) \\ \boldsymbol{0} \end{pmatrix} \\ \vdots \\ \boldsymbol{x}_{N0} - \begin{pmatrix} x_{\mathrm{q1}}(0) \\ \boldsymbol{0} \end{pmatrix} \end{pmatrix}} \tag{42}$$

are obtained. The following theorem states the second main result of this report.

Algorithm 2 *Determine the initial state $x_{\rm s0}$ of the virtual reference system*

Given: Σ_{i0}, $(i = 1, 2, ..., N)$ of the form (37)

1. Determine the matrix $\hat{\boldsymbol{A}}$ according to eqn. (40).

2. Check the synchronisation condition: $\hat{\boldsymbol{A}}$ has to be Hurwitz.

3. Determine the initial state row vector $\tilde{\boldsymbol{a}}_{\rm q1}^{\rm T}\hat{\boldsymbol{A}}^{-1}$

4. Determine the required initial state (42) .

Result: Initial state $x_{\rm s0}$ of the virtual reference system.

Theorem 3 (Initial state of the virtual reference system)
Assume that the extended agents (37) have integrator or double-integrator dynamics as in eqns. (33) and (34). If the overall systems satisfies the synchronisation condition of Theorem 1, the synchronous trajectory $y_{\rm s}(t) = x_{\rm s0}$ is constant and generated for the initial state (42).

In summary, the initial state $x_{\rm s0}$ can be determined by Algorithm 2.

5.4 Example: Synchronisation of four integrator or double-integrator agents

Consider a network of four integrator agents with Σ_{01} and Σ_{04} having a model (33) and Σ_{02} and Σ_{03} the model (34). The following parameters are used:

$$T_1 = 1,\ a_1 = 3, \quad T_2 = 0.7, \quad T_3 = 0.8, \quad T_4 = 1.2, a_4 = 4.4.$$

The networked controller is described by the parameters

$$\boldsymbol{L} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \quad \text{and} \quad k = 0.9.$$

For the initial states

$$\boldsymbol{x}_{10} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_{20} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_{30} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{x}_{40} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

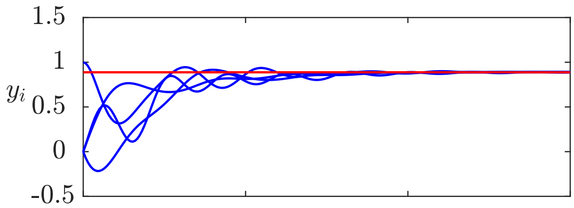

Fig. 4: Behaviour of two agents with integrator dynamics

Fig. 4 depicts the behaviour of the four networked agents.

The initial state x_{s0} of the virtual reference system is obtained by Algorithm 2. The (9×9)–matrix $\hat{\boldsymbol{A}}$ is asymptotically stable. The result

$$x_{\mathrm{s0}} = 0.8875$$

is marked by the red line in Fig. 4.

Reference

[1] Lunze, J.: *Networked Control of Multi-Agent Systems*, Edition MoRa 2022 (2nd ed.), ISBN 9789403648477.